

# Superinflation, quintessence, and the avoidance of the initial singularity

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We consider the dynamics of a spatially flat universe dominated by a self-interacting nonminimally coupled scalar field. The structure of the phase space and complete phase portraits for the conformal coupling case are given. It is shown that the non-minimal coupling modifies drastically the dynamics of the universe. New cosmological behaviors are identified, including superinflation ( $\dot{H} > 0$ ), avoidance of big bang singularities through classical birth of the universe from empty Minkowski space, and spontaneous entry into and exit from inflation. The relevance of this model to the description of quintessence is discussed.

Keywords: Inflation, quintessence.

## I. INTRODUCTION

The description of the matter content of the cosmos with a single scalar field is appropriate during important epochs of the history of the universe [1]. In this article, a dynamical system approach to a self-consistent nonsingular cosmological history is presented in the framework of the classical Einstein equations with a nonminimally coupled scalar field. The complete structure of the phase portrait and of the dynamical behavior is presented for the case of a scalar field conformally coupled to the spacetime curvature and with a quartic self-interaction potential. This exhaustive analysis is made possible thanks to a reduction of the dynamics to a two-dimensional manifold embedded in the original three-dimensional phase space, a general property shown earlier by some of the authors [9] for a classical scalar field in a spatially flat universe with arbitrary self-interaction potentials and arbitrary coupling to the curvature. Solutions with special dynamical interest are identified, namely heteroclinic and homoclinic solutions in the reduced two-dimensional phase-space, and their relevance to a possible classical birth of the universe from empty space is discussed. We recall that heteroclinic trajectories in a phase space correspond to bounded solutions connecting two different fixed points. Typically, they play the role of separatrices, determining regions of the phase space with qualitative distinct dynamical behaviors. Homoclinic trajectories, on the other hand, correspond to solutions starting and ending at the same fixed point. Their relevance to chaotic motions has been intensively discussed in the literature [2]. Despite the fact that the model presented here has no chaotic behavior, its homoclinic solutions will probably mark candidate regions of the phase space for chaotic motions if a small perturbation to the equations is introduced.

Our model consists in a universe filled with a self interacting non-minimally coupled scalar field. A crucial ingredient of the physics of scalar fields in curved spaces is their nonminimal coupling to the scalar curvature  $R$  of spacetime, which is required by first loop corrections [3], by specific particle theories [4], and by scale-invariance arguments at the classical level [5]. It is well known that nonminimal coupling dictates the success or failure of inflationary models [6]; more generally, it turns out to strongly affect the cosmic dynamics, which is qualitatively richer than in the minimally coupled case. We show, indeed, that nonminimal coupling leads to new dynamical behaviors, such as a regime that we propose to call *superinflation* ( $\dot{H} > 0$ ), which cannot be achieved with minimal coupling [7], and spontaneous entry into and exit from inflation, with or without a cosmological constant. Spontaneous superinflation provides a classical alternative to semiclassical birth of the universe from empty Minkowski space [8,9], which is impossible with minimal coupling.

In the next section, we review our model, recently proposed in [10], and give some definitions. The aspect of the phase portraits are presented in section III. Section IV is devoted to the asymptotic analyses of some especial solutions. The last section contains the concluding remarks.

## II. THE MODEL

We consider the nonminimally coupled theory described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( -\frac{R}{\kappa} + g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - 2V + \xi R \psi^2 \right), \quad (1)$$

where  $R$  denotes the scalar curvature,  $\psi$  is the scalar field,  $\kappa \equiv 8\pi G$  ( $G$  being Newton's constant), and  $\xi$  is the nonminimal coupling constant. A cosmological constant  $\Lambda$ , if present, is incorporated in the scalar field potential  $V(\psi)$ . We use the full conserved scalar field stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \xi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) (\psi^2) + \xi G_{\mu\nu} \psi^2 - \frac{1}{2} g_{\mu\nu} (\partial_\alpha \psi \partial^\alpha \psi - 2V(\psi)) \quad (2)$$

(where  $G_{\mu\nu}$  is the Einstein tensor), thereby avoiding the widespread effective coupling  $\kappa_{\text{eff}} = \kappa (1 - \kappa \xi \psi^2)^{-1}$  in the Einstein equations  $G_{\mu\nu} = \kappa T_{\mu\nu}$ . We consider here the dynamics of a spatially flat Friedmann-Robertson-Walker universe with line element  $ds^2 = d\tau^2 - a^2(\tau) (dx^2 + dy^2 + dz^2)$ . This yields the trace equation  $R = -\kappa (\sigma - 3p)$ , the energy constraint  $3H^2 = \kappa \sigma$  (which guarantees that the energy density  $\sigma \geq 0$ ), and the Klein-Gordon equation. More explicitly,

$$6 [1 - \xi (1 - 6\xi) \kappa \psi^2] \left( \dot{H} + 2H^2 \right) - \kappa (6\xi - 1) \dot{\psi}^2 - 4\kappa V + 6\kappa \xi \psi \frac{dV}{d\psi} = 0, \quad (3)$$

$$\frac{\kappa}{2} \dot{\psi}^2 + 6\xi \kappa H \psi \dot{\psi} - 3H^2 (1 - \kappa \xi \psi^2) + \kappa V = 0, \quad (4)$$

$$\ddot{\psi} + 3H \dot{\psi} + \xi R \psi + \frac{dV}{d\psi} = 0. \quad (5)$$

The (time-dependent) equation of state of the  $\psi$  field, rather than being imposed *a priori*, follows self-consistently from the dynamics. In the trace equation (3), the second derivative  $\ddot{\psi}$  that appears in the pressure has been replaced by its expression given by the Klein-Gordon equation. Clearly, in the set of equations (3)-(5), the sub-system (3)-(4) is a closed implicit two-dimensional system for  $\psi$  and  $H = \dot{a}/a$  (note that this dimensional reduction is not possible for spatially curved universes [11,9]). After solving these implicit equations for  $\dot{\psi}$  and  $\dot{H}$ , one has

$$\dot{\psi} = -6\xi H \psi \pm \frac{1}{2\kappa} \sqrt{\mathcal{G}(H, \psi)}, \quad (6)$$

$$\dot{H} = \frac{1}{1 + \kappa \xi (6\xi - 1) \psi^2} \left[ 3(2\xi - 1) H^2 + 3\xi (6\xi - 1) (4\xi - 1) \kappa H^2 \psi^2 \mp \xi (6\xi - 1) H \psi \sqrt{\mathcal{G}} + (1 - 2\xi) \kappa V(\psi) - \kappa \xi \psi \frac{dV}{d\psi} \right], \quad (7)$$

where

$$\mathcal{G}(H, \psi) = 8\kappa^2 \left[ \frac{3H^2}{\kappa} - V(\psi) + 3\xi (6\xi - 1) H^2 \psi^2 \right]. \quad (8)$$

Due to the energy constraint (4), the trajectories are restricted to a two-dimensional manifold  $\Sigma$  in the three-dimensional  $(H, \psi, \dot{\psi})$  original phase space, possibly with “holes” (dynamically forbidden regions) corresponding to  $\mathcal{G}(H, \psi) < 0$  (cf. Eq. (6)).  $\Sigma$  is composed of two sheets corresponding to the positive or negative sign in Eq. (6). The two sheets smoothly join on the boundary  $\mathcal{G} = 0$  of the dynamically forbidden region. In the present work, we restrict to the potential

$$V(\psi) = \frac{3\alpha}{\kappa} \psi^2 - \frac{\Omega}{4} \psi^4 - \frac{9\omega}{\kappa^2}, \quad (9)$$

consisting of a mass term, a quartic self-coupling and, possibly, a cosmological constant term. For consistency with previous works [9] we use the symbols  $\alpha \equiv \kappa m^2/6$  ( $m$  being the scalar field mass) and  $\omega \equiv -\kappa^2 \Lambda/9$ . Figures 1 and 2 present some aspects of  $\Sigma$  for the potential (9). However, several of our main results do not depend on the details of  $V(\psi)$ .

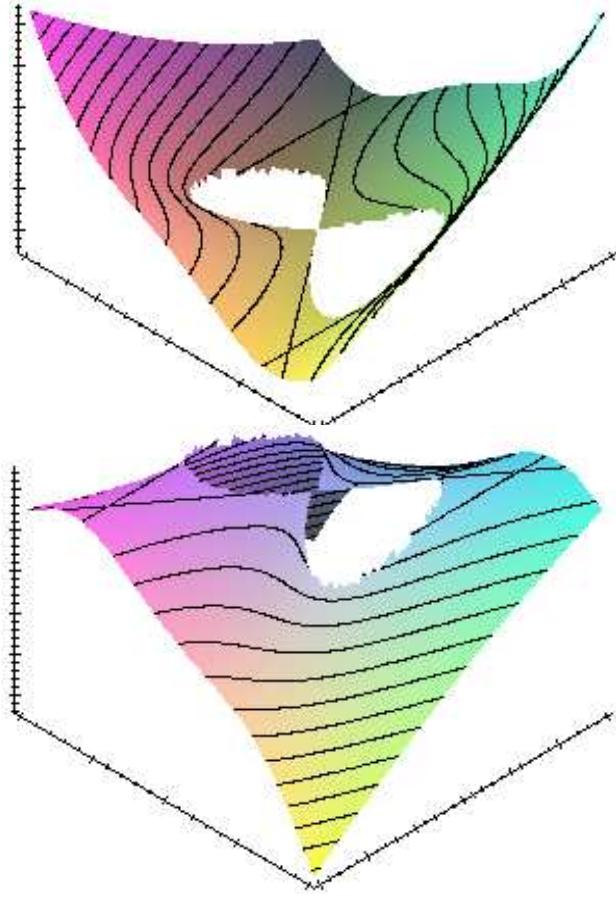


FIG. 1. Aspect of the two-dimensional manifold  $\Sigma$ , embedded in the three-dimensional phase space  $(\psi, H, \dot{\psi})$ , for the potential (9). The upper graph represents the “+” sheet, while the lower one depicts the “-” sheet. In fact, they are not disconnected, the two sheets join smoothly on the boundary  $\mathcal{G}=0$  of the dynamically forbidden region, corresponding to the showed “holes”. The lines of constant  $\mathcal{G}$  are drawn on the sheets.

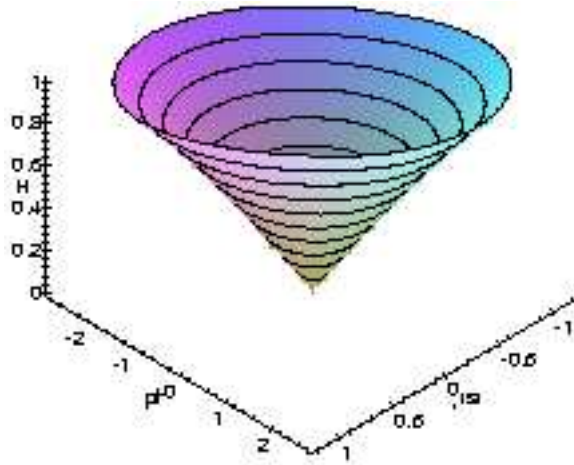


FIG. 2. Zoom, near the origin, of the two-dimensional manifold  $\Sigma$ , embedded in the three-dimensional phase space  $(\psi, \dot{\psi}, H)$ , for the potential (9). The two sheets join forming two symmetric cones with their apex at the origin. Only the cone corresponding to  $H > 0$  is presented here, and the lines of constant  $H$  are drawn.

### III. PHASE SPACE PORTRAITS

In the following, for simplicity, we project the dynamics of the phase space onto the  $(H, \psi)$  plane, but the true nature of  $\Sigma$  should always be kept in mind. The fixed points of the system (3)-(5) include de Sitter solutions with constant scalar field

$$H_0^2 = \frac{3(\alpha^2 - \Omega\omega)}{\kappa(\Omega - 6\xi\alpha)}, \quad \psi_0^2 = \frac{6(\alpha - 6\xi\omega)}{\kappa(\Omega - 6\xi\alpha)} \quad (10)$$

( $\Omega \neq 6\alpha\xi$ ), and the solutions  $(H, \psi) = (\pm\sqrt{-3\omega/\kappa}, 0)$ . The fixed points (10) exist also for  $\omega = 0$ , due to the presence of the matter field  $\psi$  (in this case, the two points  $(\pm\sqrt{-3\omega/\kappa}, 0)$  collapse into the Minkowski space fixed point, which is at the apex of the cones described in Figure 2). Here, we restrict ourselves to the case of conformal coupling,  $\xi = 1/6$ .

The function

$$L(\psi, \dot{\psi}) = \frac{1}{2}\dot{\psi}^2 + \frac{\alpha}{4}\psi^4 - \frac{3\omega}{\kappa}\psi^2 + V(\psi) \quad (11)$$

is such that  $dL/dt = -3H\dot{\psi}^2$  along the trajectories. For  $H > 0$ ,  $L$  is a Lyapunov function in a region containing the origin; the solutions are then confined by closed lines of constant  $L$ , implying asymptotic convergence to the fixed points on the  $H$  axis (from Eq. (4), if  $\psi$  and  $\dot{\psi}$  vanish,  $H$  goes to  $V(0)$ ). This behavior is confirmed by exhaustive numerical simulations [10] and reported in the following. We first exclude a cosmological constant by setting  $\omega = 0$ . The phase portrait qualitatively differs according to the ratio  $\Omega/\alpha$ .

#### A. The case $\Omega = 2\alpha$

The Minkowski space  $(H, \psi, \dot{\psi}) = (0, 0, 0)$  is a fixed point, attractive for  $H > 0$  and repulsive for  $H < 0$ ; the projections of the de Sitter spaces  $(\pm H_0, \pm\psi_0, 0)$  are saddle points, i.e. they possess attractive and repulsive eigendirections in the phase space (Fig. 3a). They are of two kinds: expanding ( $H\psi > 0$ ) or contracting ( $H\psi < 0$ ). The following solutions, present only in this particular case,

$$H(\tau) = \sqrt{\frac{C}{2}} \tanh\left(\sqrt{2C}\tau\right), \quad \psi = \pm\psi_0 \equiv \pm\sqrt{\frac{6}{\kappa}} \quad (12)$$

(where  $C = \dot{H} + 2H^2 = -R/6$  is constant), correspond to heteroclinic straight lines connecting de Sitter fixed points, starting along the repulsive eigendirection of one of them and ending along the attractive eigendirection of the other (Fig. 3a). They are tangent to the boundary of the forbidden regions at  $(H, \psi) = (0, \pm\psi_0)$ . For  $|H| > \sqrt{C/2}$ , another straight line solution is obtained from the general form

$$H(\tau) = \sqrt{\frac{C}{2}} \frac{w_1 e^{\sqrt{C/2}\tau} - w_2 e^{-\sqrt{C/2}\tau}}{w_1 e^{\sqrt{C/2}\tau} + w_2 e^{-\sqrt{C/2}\tau}}, \quad \psi = \psi_0, \quad (13)$$

where  $w_1$  and  $w_2$  are integration constants. The nonsingular solutions (12) connect a contracting ( $\tau \rightarrow -\infty$ ) de Sitter regime to a minimum nonvanishing value of the scale factor ( $\tau = 0$ ), and then to an expanding de Sitter regime ( $\tau \rightarrow +\infty$ ).

In addition to these straight lines, we found numerically other heteroclinic solutions: one starting at  $(H, \psi) = (0, 0)$  and ending at  $(-H_0, \psi_0)$ , and another one from  $(H_0, \psi_0)$  to  $(0, 0)$ . A third solution starts at the expanding de Sitter point and goes to infinity, while another one comes from infinity and arrives to the contracting de Sitter point. The phase portrait is symmetric about the origin.

Near the fixed point  $(0, 0, 0)$ , numerical analysis confirms the peculiar behavior suggested by the Lyapunov function: orbits approaching this point with positive  $H$  are attracted to it, bouncing back and forth infinitely many times off the  $\mathcal{G} = 0$  boundary in the  $(H, \psi)$  projection (Fig. 3a). In the space  $(H, \psi, \dot{\psi})$  these orbits are seen to spiral down on a cone towards its apex at the origin. The cone results from the union of the two sheets in the vicinity of the origin, as it is shown in Figure 3. Along the spiral, the orbit passes almost periodically from one sheet to the other with period  $\tau_{\text{bounce}} = 2\pi/m$  ( $\tau_{\text{bounce}}$  is obtained from the asymptotic analysis of the next section. Typically, after a few bounces, the period coincides with  $\tau_{\text{bounce}}$  with good accuracy). A similar behavior for  $\Omega < 0$  was reported in the earlier numerical analysis of [11], but using the effective coupling  $\kappa_{\text{eff}}(\tau)$  and the variables  $\psi$  and  $\dot{\psi}$ .

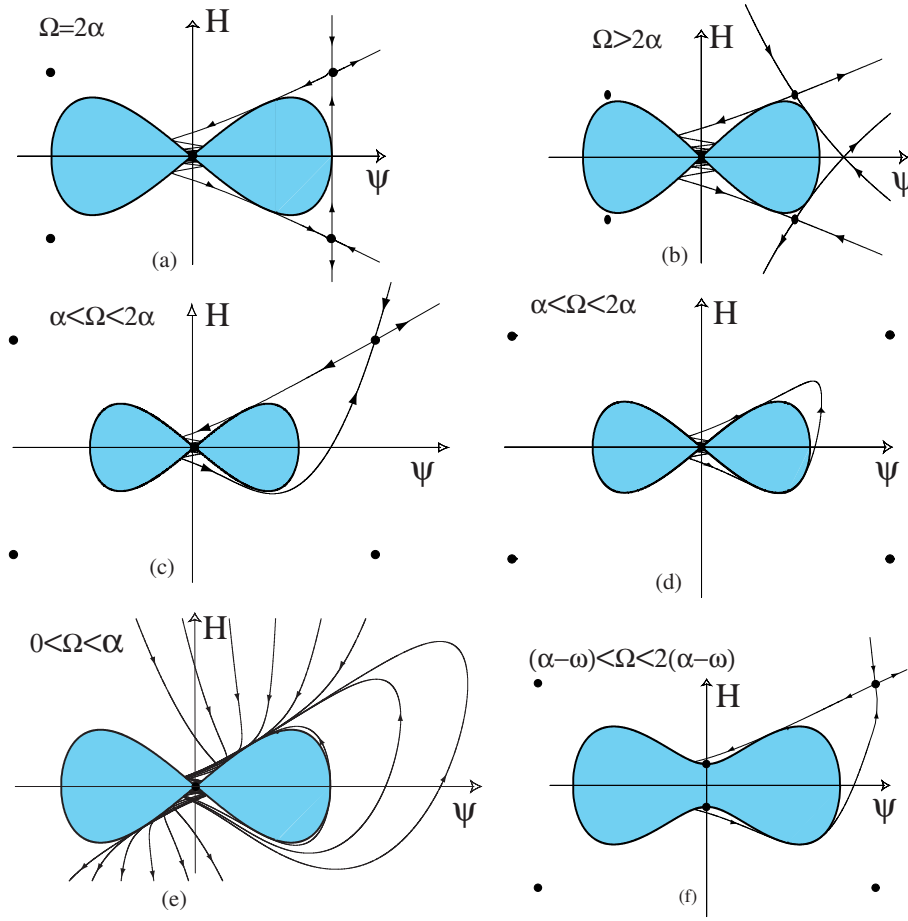


FIG. 3. Qualitative phase portraits for the system (3)-(5). Shaded regions correspond to the dynamically forbidden regions ( $\mathcal{G}(H, \psi) < 0$ , cf. Eq. (6)). Figures a-e, were obtained by using  $\Omega/\alpha$ , respectively, 2, 5,  $3/2$ ,  $3/2$ , and  $1/2$ , and  $\omega = 0$ . Figure f corresponds to the case  $\omega = 1/10$  and  $\Omega/(\alpha - \omega) = 3/2$ .

In the  $H < 0$  half-plane, the situation is reversed: orbits starting with  $H < 0$  are repelled by the origin and depart from it bouncing off the  $\mathcal{G} = 0$  boundary.

### B. The case $\Omega > 2\alpha$

The situation (Fig. 3b) is analogous to the previous one, but now the straight heteroclinic solutions are missing, and are replaced by the solution starting at the contracting de Sitter fixed point and escaping to infinity, and by the solution coming from infinity and arriving to the expanding de Sitter point. The two-sheeted structure of  $\Sigma$  implies that no actual intersections occur between different orbits in Fig. 3b, which live in different sheets but are projected on the same plane.

### C. The case $\alpha < \Omega < 2\alpha$

As shown in Fig. 3c, there are no straight heteroclinic lines but new interesting features emerge. A new heteroclinic solution appears starting from the origin and ending in the expanding de Sitter fixed point. As in the previous case, the quadrant  $\psi > 0, H < 0$  is obtained from the  $\psi > 0, H > 0$  one by reflection about the  $\psi$ -axis and time-reversal in Fig. 3c.

The crucial feature of this case is the appearance of a dense set of homoclinic solutions (Fig. 3d) departing from the origin with negative  $H$  and returning to it with positive  $H$ , going around the forbidden region. Superinflation plays a central role along these orbits; only a regime with  $\dot{H} > 0$  permits the smooth transition from an initial contracting

( $H < 0$ ) phase to an expanding one ( $H > 0$ ). This transition occurs at the nonvanishing minimum of the scale factor. The behavior of this family of homoclinics, as well as of the other solutions, is universal: they rapidly converge in the spiraling region near the origin, irrespective of initial conditions. Since all these homoclinics originate from a neighborhood of the Minkowski fixed point due to its own instability with respect to perturbations with  $H < 0$ , and come back to that point, due to the stability for perturbations with  $H > 0$ , this behavior constitute a classical alternative to the previously proposed semiclassical birth of the universe from empty space [8,9].

#### D. The case $0 < \Omega < \alpha$

The fixed points (10) disappear and the only bounded solutions are the homoclinics associated with the origin (see Fig. 3e). This situation is therefore the most favorable for the classical spontaneous exit from empty Minkowski space.

#### E. The case $\omega < 0$

Analogous results hold if a small cosmological constant is present (see Figs. 3f), with the phase portrait being classified according to  $\Omega/(\alpha - \omega)$ , but the fixed point  $(0, 0, 0)$  of the  $\omega = 0$  case splits into two de Sitter fixed points with memory of the previous stability properties. Now, the approximate period between two consecutive bounces is

$$\tau_{\text{bounce}} = \frac{2\pi}{\sqrt{m^2 + \frac{3\omega}{4\kappa}}} . \quad (14)$$

### IV. ASYMPTOTIC ANALYSIS

For the conformally coupled case, the trace and the Klein Gordon equations (3) and (5), after a rescaling and the explicit substitution of the Hamiltonian constraint (4), read

$$\begin{aligned} \dot{H} + 2H^2 - \alpha\psi^2 + 6\omega &= 0 \\ \ddot{\psi} + 3H\dot{\psi} + 6(\alpha - \omega)\psi - (\Omega - \alpha)\psi^3 &= 0 \end{aligned} \quad (15)$$

We have special interest in the asymptotic behavior of the solutions of (15) in the vicinity of the attractive fixed points. Let us start with the case  $\omega = 0$ . In this case, our region of interest is the neighborhood with  $H > 0$  of the origin. We search for asymptotic solutions of the form

$$\begin{aligned} \psi(t) &= \frac{f_1(t)}{t} + \frac{f_2(t)}{t^2} + \mathcal{O}(t^{-3}) \\ H(t) &= \frac{g_1(t)}{t} + \frac{g_2(t)}{t^2} + \mathcal{O}(t^{-3}) \end{aligned} \quad (16)$$

with  $f_1(t), f_2(t), g_1(t)$ , and  $g_2(t)$  bounded for large  $t$ . Inserting (16) in the equations (15) one has

$$\begin{aligned} \frac{1}{t} (f_1'' + 6\alpha f_1) + \frac{1}{t^2} (f_2'' - 2f_1' + 3g_1 f_1' + 6\alpha f_2) &= \mathcal{O}(t^{-3}) \\ \frac{1}{t} g_1' + \frac{1}{t^2} (g_2' - g_1 + 2g_1^2 - \alpha f_1^2) &= \mathcal{O}(t^{-3}) \end{aligned} \quad (17)$$

By demanding the exactness of (17) up to  $t^{-3}$  order, one gets immediately that  $g_1$  is a constant and  $f_1 = A \cos(\sqrt{6\alpha}t + \delta)$ , leading to

$$f_2'' + 6\alpha f_2 = A\sqrt{6\alpha}(3g_1 - 2) \sin(\sqrt{6\alpha}t + \delta), \quad (18)$$

which has the general solution

$$f_2 = B \cos(\sqrt{6\alpha}t + \theta) - \frac{A(3g_1 - 2)}{2\sqrt{6\alpha}} \left( \cos(\sqrt{6\alpha}t + \delta) \sqrt{6\alpha}t - \cos(\sqrt{6\alpha}t) \sin \delta \right). \quad (19)$$

In order to guarantee the boundedness of  $f_2$ , it is necessary to have  $g_1 = 2/3$ . From the equation corresponding to the  $t^{-2}$  term in the trace equation, we have

$$g_2' = A^2 \alpha \cos^2(\sqrt{6\alpha}t + \delta) - \frac{2}{9}, \quad (20)$$

which is solved by

$$g_2 = \frac{A^2 \alpha}{2\sqrt{6\alpha}} \left( \cos(\sqrt{6\alpha}t + \delta) \sin(\sqrt{6\alpha}t + \delta) + \delta \right) + \left( \frac{A^2 \alpha}{2} - \frac{2}{9} \right) t. \quad (21)$$

Again, by requiring the boundedness of  $g_2$  for large  $t$ , we get the condition  $A = 2/(3\sqrt{\alpha})$ . We have, finally, the following asymptotic solution for the  $\omega = 0$  case

$$\begin{aligned} \psi(t) &= \frac{2 \cos \sqrt{6\alpha}t}{3\sqrt{\alpha}t} + \mathcal{O}(t^{-2}) \\ H(t) &= \frac{2}{3t} + \mathcal{O}(t^{-2}). \end{aligned} \quad (22)$$

From (22) we can obtain the characteristic period  $\tau_{\text{bounce}} = 2\pi/\sqrt{6\alpha}$ . Also, the asymptotic solution (22) implies that all solutions ending in the origin will behave as matter dominated universes for large  $t$ , corresponding to  $a(t) \propto t^{2/3}$ .

The case with a small cosmological constant can be treated analogously, taking into account that now the relevant fixed point is  $\psi = 0$  and  $H = \sqrt{-3\omega}$ . The search for asymptotic solutions of the form

$$\begin{aligned} \psi(t) &= \frac{f_1(t)}{t} + \frac{f_2(t)}{t^2} + \mathcal{O}(t^{-3}) \\ H(t) &= g_0(t) + \frac{g_1(t)}{t} + \frac{g_2(t)}{t^2} + \mathcal{O}(t^{-3}) \end{aligned} \quad (23)$$

with  $f_1(t), f_2(t), g_0(t), g_1(t)$ , and  $g_2(t)$  bounded for large  $t$ , as before, give rise to the equations

$$\begin{aligned} \frac{1}{t} (f_1'' + 3g_0 f_1' + 6(\alpha - \omega)f_1) + \frac{1}{t^2} (f_2'' - 2f_1' + 3(g_1 f_1' + g_0 f_2' - g_0 f_1) + 6(\alpha - \omega)f_2) &= \mathcal{O}(t^{-3}), \\ (g_0' + 2g_0 + 6\omega) + \frac{1}{t} (g_1' + 4g_0 g_1) + \frac{1}{t^2} (g_2' - g_1 + 4g_0 g_1 + 2g_1^2 - \alpha f_1^2) &= \mathcal{O}(t^{-3}). \end{aligned} \quad (24)$$

Our asymptotic equations now are considerably more complicated, but, nevertheless, we can obtain an unambiguous period  $\tau_{\text{bounce}}$ . From (24), one has that  $g_0$  decreases exponentially to the value  $\sqrt{-3\omega}$  for large  $t$ , implying that  $f_1$  converge to the form  $f_1 = A \exp(-3\sqrt{-3\omega}t/2) \cos(\sqrt{6\alpha + 3\omega/4}t + \delta)$ , from which one can obtain  $\tau_{\text{bounce}} = 2\pi/\sqrt{6\alpha + 3\omega/4}$ . The scale factor, in this case, obeys  $a(t) \propto \exp \sqrt{-3\omega}t$  for large  $t$ , as in the de Sitter universe.

## V. DISCUSSIONS

The  $(H, \psi)$  plane is divided into sectors by the straight lines  $H = \pm\sqrt{\alpha/2}\psi$ ,  $H = \pm\sqrt{\alpha}\psi$ ,  $H = \pm\sqrt{2\alpha}\psi$  corresponding, respectively, to  $\dot{H} = 0$ ,  $\ddot{a} = 0$ , and pressure  $p = 0$  (the  $H$ -axis corresponds to  $p = \sigma/3$ ). The lines  $H = \pm\sqrt{\alpha/2}\psi$  mark the transition between inflationary ( $\ddot{a} > 0$  and  $\dot{H} \leq 0$ ) and superinflationary ( $\dot{H} > 0$ ) regimes. The lines  $H = \pm\sqrt{\alpha}\psi$  divide regions corresponding to inflation and to decelerated expansion, while  $H = \pm\sqrt{2\alpha}\psi$  divide regions of positive and negative pressures (Fig. 4). The crucial condition for superinflation to occur is that the line  $\dot{H} = 0$  (or parts of it) should belong to the dynamically accessible region  $\mathcal{G} \geq 0$  of the  $(H, \psi)$  plane. This implies that, for an arbitrary potential  $V(\psi)$ , superinflation corresponds to  $\psi dV/d\psi \leq 0$  (this result will be discussed in detail elsewhere). For our particular potential (9), this requires  $\Omega > 0$ . The superinflationary behavior occurs only once along each homoclinic and brings the solution from the primordial Minkowski neighborhood to the succession of eras corresponding to different equations of state (during each bounce), towards infinite dilution and equation of state  $p = 0$ . In fact, asymptotic analysis for  $\tau \rightarrow +\infty$  and for any value of  $\alpha$  and  $\Omega$  (with  $\omega = 0$ ) shows that the scale factor  $a(\tau)$  exhibits oscillations of concavity corresponding to accelerated and decelerated epochs. These oscillations are damped as  $\tau \rightarrow +\infty$ ; in this regime,  $a(\tau) \propto \tau^{2/3}$  and the universe becomes matter dominated.

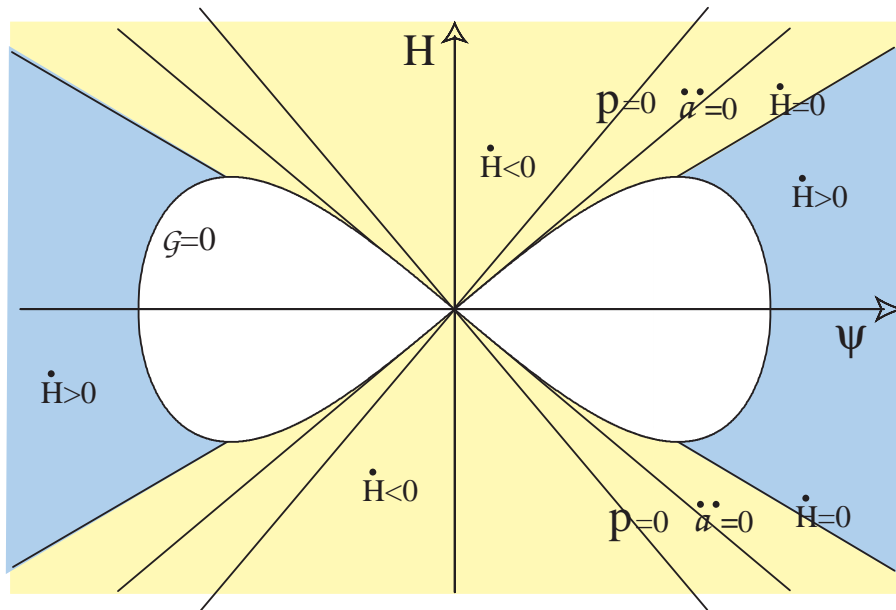


FIG. 4. The plane  $(H, \psi)$  for the  $\omega = 0$  case and the equation of state for  $\psi$ . The darkest region corresponds to the superinflation regime ( $\dot{H} > 0$ ). During the bounces of the  $\psi$  solution in the region  $\dot{H} < 0$ , its equation of state corresponds to, respectively, radiation domination (crossing the  $H$ -axis), matter domination ( $p = 0$ ), and re-acceleration (between  $\mathcal{G} = 0$  and  $\ddot{a} = 0$  lines). These oscillations are damped as  $\tau \rightarrow +\infty$ , the universe becomes matter dominated  $a(\tau) \propto \tau^{2/3}$  and tends to infinite dilution.

While it is not claimed here that the evolution of our universe is modeled by an entire orbit of the system (3)-(5) on the accessible manifold  $\Sigma$ , the application to specific eras of the cosmological history is intriguing. Indeed, in the bounces reported above (during which  $\dot{H} < 0$ ), one encounters, respectively, radiation domination crossing the  $H$ -axis, matter domination ( $p = 0$ ), acceleration (a possible quintessence model?) until the next bounce in the  $(H, \psi)$  projection, where this sequence is reversed.

If we identify one period as our cosmological history, then the reported accelerated expansion of the universe today [13] suggests to locate our epoch in the sector between the line  $H = \sqrt{\alpha} \psi$  and the  $\mathcal{G} = 0$  boundary. The identification of the age of the universe ( $\sim 10^{17}$  s) with  $\tau_{\text{bounce}}$  would then yield the scalar field mass  $m \simeq 10^{-13}$  eV, which is suggestive of an axion [1] or of an ultralight pseudo-Goldstone boson (that the quintessence field should be very light was already suggested [14]).

As a step towards a realistic model, one can include a second scalar field coupled to  $\psi$ , which has the meaning of a fundamental field (as is done, e.g., in hybrid inflation), or mimics a baryonic or other fluid. In spite of the higher dimensionality of the phase space, many of the features exposed here for a single scalar field survive [15]. Although very simple, we think our classical model opens interesting avenues to the understanding of quintessence in terms of a non-minimally coupled scalar field.

Finally, one should keep in mind that our detailed semi-analytical analyses are possible thanks to the reduction of the original three-dimensional system to a two-dimensional one on a smooth manifold  $\Sigma$ , which, in turn, strongly indicates the absence of chaos (confirmed by an exhaustive numerical analysis). Generically, such a reduction is not possible for the case of a spatially curved spacetime and/or several matter fields non-minimally coupled to the curvature, despite of many similar results. This is object of present investigations [15].

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